Gaussian Elimination

$$3x + 4y = 7 \qquad ①$$
$$2x + 3y = 5 \qquad ②$$

Augmented

$$\begin{array}{cc} A & \vec{b} \\ \end{array}$$
$$\left[\begin{array}{cc:c} 3 & 4 & 7 \\ 2 & 3 & 5 \end{array}\right] \qquad A\vec{x} = \vec{b}$$

$\frac{1}{2}\cdot ①$ $\qquad x + \frac{4}{3}y = \frac{7}{3} \qquad ③$

$$\tfrac{1}{3}\vec{r}_1 \to \vec{r}_1 \begin{bmatrix} 1/3 & 0 \\ 0 & 1 \end{bmatrix}\left[\begin{array}{cc|c} 3 & 4 & 7 \\ 2 & 3 & 5 \end{array}\right] = \left[\begin{array}{cc|c} 1 & 4/3 & 7/3 \\ 2 & 3 & 5 \end{array}\right]$$

$② - 2\cdot③$

$$\begin{aligned} &2x + 3y = 5 \\ -&(2x + \tfrac{4}{3}y) = -2\cdot\tfrac{7}{3} \end{aligned}$$

$$-2\vec{r}_1 + \vec{r}_2 \to \vec{r}_2 \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}\left[\begin{array}{cc:c} 1 & 4/3 & 7/3 \\ 2 & 3 & 5 \end{array}\right] = \left[\begin{array}{cc:c} 1 & 4/3 & 7/3 \\ 0 & 1/3 & 1/3 \end{array}\right]$$

$\Leftrightarrow \quad \frac{1}{3}y = \frac{1}{3}$

$\Leftrightarrow \quad y = 1 \qquad ④$

$$\Updownarrow$$

$④ \to ③ \quad x = 1$

$$\begin{bmatrix} 1 & 4/3 \\ 0 & 1/3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 7/3 \\ 1/3 \end{bmatrix} \qquad (*)$$

upper Δ

$\begin{bmatrix} 1/3 & 0 \\ 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}$ elementary row operations

---

Type 1:

$$\begin{bmatrix} 1 & & & & & 0 \\ & \ddots & & & & \\ & & 1 & & & \\ & & & \alpha & & \\ & & & & \ddots & \\ 0 & & & & & 1 \end{bmatrix}$$

$\uparrow i$

$\alpha\vec{r}_i \to \vec{r}_i \qquad \alpha > 0$

Type 2:

$$\begin{bmatrix} 1 & & & & \\ & \ddots & & & \\ & -\alpha & 1 & & \\ & & & \ddots & \\ & & & & 1 \end{bmatrix} = \begin{bmatrix} 1 & & & & \\ & \ddots & & & \\ & +\alpha & 1 & & \\ & & & \ddots & \\ & & & & 1 \end{bmatrix}^{-1}$$

$\uparrow j \quad \uparrow i$

$\vec{r}_i - \alpha\vec{r}_j \to \vec{r}_i$

Both are invertible

Solving $A\vec{x} = \vec{b} \Leftrightarrow \underbrace{E_\ell \cdots E_1 A}_{U}\vec{x} = \underbrace{E_\ell \cdots E_1 \vec{b}}_{\vec{c}}$

s.t. $U\vec{x} = \vec{c}$

where U is upper triangular (see $(*)$)

Then $\vec{x}$ can be obtained by back substitution.

$$\text{BFS} \xrightarrow{\text{ERO's}} \text{BFS}$$

Eg 3.1 Observations

(1) ERO recorded in last n columns.

$$A = (R_1 \mid \underset{}{\overset{B_1}{\overset{\shortparallel}{I}}})$$

$$\Updownarrow$$

$$\underbrace{E_\ell \cdots E_1}_{E} A = (U_i \mid V_i) \text{ tableau } i$$

$$\Updownarrow$$

$$EA = (U_i \mid V_i)$$

$$\Updownarrow$$

$$E(R_1 \mid I) = (U_i, V_i) \implies E = V_i$$

Eg Tableau 3

$$\left(\begin{array}{ccc:ccc} 1 & 0 & -2/5 & 3/5 & 1/5 & 0 \\ 0 & 1 & -3/5 & 2/5 & -1/5 & 0 \\ 0 & 0 & -1/5 & -1/5 & 3/5 & 1 \end{array}\right) T_3$$

$$= \underbrace{E_\ell \cdots E_1}_{E} \left(\begin{array}{ccc:ccc} 1 & 1 & -1 & 1 & 0 & 0 \\ 2 & 3 & 1 & 0 & 1 & 0 \\ -1 & 2 & -1 & 0 & 0 & 1 \end{array}\right) T_1$$

$$\implies E = \boxed{\begin{pmatrix} 3/5 & 1/5 & 0 \\ 2/5 & -1/5 & 0 \\ -1/5 & 3/5 & 1 \end{pmatrix}}$$

(ii) B_i are columns of A corresponding to $\vec{e}_j$ in tableau i.

Tableau i

$$\left(\times\times \vec{e}_{i_1} \times\times \ \vec{e}_{i_2} \times \cdots \cdots \times \ \vec{e}_{i_n} \times \cdots \right)$$

Since $\underbrace{E_\ell \cdots E_1}_{E} A = \left(\times\times \ \vec{e}_{i_1} \times\times \ \vec{e}_{i_2} \times \cdots \times \ \vec{e}_{i_n} \times \cdots \right)$

$$A = E^{-1} \left(\times\times \ \vec{e}_{i_1} \times \cdots \vec{e}_{i_2} \cdots \cdots \vec{e}_{i_n} \times \cdots \right)$$

$$\Rightarrow \quad E^{-1} = \left(\vec{a}_{i_1} \cdot \vec{a}_{i_2} \cdots \vec{a}_{i_n} \right)$$

Since $B_i = \left(\vec{a}_{i_1} \cdots \ \vec{a}_{i_n} \right)$

$$\therefore \quad A = B_i \underbrace{\left(\times\times \vec{e}_{i_1} \times \cdots \vec{e}_{i_2} \cdots \cdots \times \vec{e}_{i_n} \times \cdots \right)}_{\Vert}$$

$$\Rightarrow \qquad Y_i$$

Eg. Tableau 3 $\begin{array}{cccccc} x_1 & x_2 & & & & x_6 \end{array}$ $\begin{pmatrix} 1 & 0 & \times & \times & \times & 0 \\ 0 & 1 & \times & \times & \times & 0 \\ 0 & 0 & \times & \times & \times & 1 \end{pmatrix}$ Current basic soln is x_1, x_2, x_6

$$\underbrace{E_\ell \cdots E_1}A = \begin{pmatrix} 1 & 0 & & 0 \\ 0 & 1 & \times & 0 \\ 0 & 0 & & 1 \end{pmatrix} \Rightarrow A = E^{-1} \begin{pmatrix} 1 & 0 & & 0 \\ 0 & 1 & \times & 0 \\ 0 & 0 & & 1 \end{pmatrix}$$

$$\Rightarrow E^{-1} = \left(\vec{a}_1, \vec{a}_2, \vec{a}_6 \right) = B_3$$

as x_1, x_2, x_6 are basic variables

(iii) If $E_l \cdots E_1 [A \mid \vec{b}] = [\bar{I}_i \mid \vec{y}_0]$

then $\vec{y}_0 = B_i^{-1}\vec{b} = \vec{X}_{B_i}$ the basic solution corresponding to basic matrix B_i

Pf: $\underbrace{E_l \cdots E_1}_{E} [A \mid \vec{b}] = [\bar{I}_i \mid \vec{y}_0]$

$\Rightarrow \quad E\vec{b} = \vec{y}_0$

$\Rightarrow \quad \vec{y}_0 = B_i^{-1}\vec{b}$ (as $E^{-1} = B_i$ observation (ii))

Recall. $A\vec{x} = (B_i \mid R_i)\begin{pmatrix} \vec{X}_{B_i} \\ \vec{0} \end{pmatrix} = \vec{b}$

$\Rightarrow \quad B_i \vec{X}_{B_i} = \vec{b} \Rightarrow \vec{x}_{B_i} = B_i^{-1}\vec{b} = \vec{y}_0$ #

Eg.

$$\left(\underbrace{\begin{matrix} 1 & 1 & -1 & 1 & 0 & 0 \\ 2 & -3 & 1 & 0 & 1 & 0 \\ -1 & 2 & -1 & 0 & 0 & 1 \end{matrix}}_{A} \;\vdots\; \underbrace{\begin{matrix} 5 \\ 3 \\ 1 \end{matrix}}_{\vec{b}}\right) \rightsquigarrow \begin{array}{c} \begin{matrix} x_1 & x_2 & & & & x_6 \end{matrix} \\ \left(\underbrace{\begin{matrix} 1 & 0 & \times & \times & \times & 0 \\ 0 & 1 & \times & \times & \times & 0 \\ 0 & 0 & \times & \times & \times & 1 \end{matrix}}_{\bar{I}_3} \;\vdots\; \underbrace{\begin{matrix} 18/5 \\ 7/5 \\ 9/5 \end{matrix}}_{\vec{y}_0}\right) \end{array} \begin{matrix} \leftarrow x_1 \\ \leftarrow x_2 \\ \leftarrow x_6 \end{matrix}$$

$$\Rightarrow \underbrace{\begin{pmatrix} 1 & 1 & -1 & 1 & 0 & 0 \\ 2 & -3 & 1 & 0 & 1 & 0 \\ -1 & 2 & -1 & 0 & 0 & 1 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} 18/5 \\ 7/5 \\ 0 \\ 0 \\ 0 \\ 9/5 \end{pmatrix}}_{\vec{x}} = \underbrace{\begin{pmatrix} 5 \\ 3 \\ 1 \end{pmatrix}}_{\vec{b}} \quad \#$$

In fact $A\vec{x} = B_3\bar{I}_3\vec{x} = B_3(\vec{e}_1\ \vec{e}_2\ \vec{*}\ \vec{*}\ \vec{*}\ \vec{e}_3)\begin{pmatrix} 18/5 \\ 7/5 \\ 0 \\ 0 \\ 0 \\ 9/5 \end{pmatrix}$

$= B_3 I_3 \begin{pmatrix} 18/5 \\ 7/5 \\ 9/5 \end{pmatrix} = B_3\vec{y}_0 = B_3 B_3^{-1} b = \vec{b}$

$(\because E\vec{b} = \vec{y}_0$

$\Rightarrow B_3^{-1}\vec{b} = \vec{y}_0)$

Gaussian Elimination in Simplex Algorithm

$$A = [\, N_1 \,\vdots\, \underbrace{\begin{matrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{matrix}}_{B_1} \,] \qquad B_1 = \begin{bmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{bmatrix} = [\vec{e}_1, \vec{e}_2, \dots, \vec{e}_m] = I$$

$$= B_1 [\, B_1^{-1} N \mid I \,]$$

$$= B_1 Y_1$$

Suppose x_{B_r} is leaving and x_j is entering, i.e. replace $\vec{e}_r$ by $\vec{a}_j$

If by ERO:

$$\begin{pmatrix} 0 \\ \vdots \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \leftarrow r$$

$$\underbrace{E_\ell \cdots E_1}_{C^{-1}} \cdot A = [\vec{u}_1, \vec{u}_2, \cdots \vec{u}_{j-1}, \underset{\uparrow\, j}{\vec{e}_r}, \vec{u}_{j+1} \cdots \vec{u}_{n-m} \,\vdots\, \vec{e}_1 \cdots \vec{e}_{r-1}, \underset{\uparrow\, r}{\vec{u}_r}, \vec{e}_{r+1} \cdots \vec{e}_m] \quad (1)$$

Note that $B_2 = [\vec{e}_1, \vec{e}_2, \dots, \vec{e}_{r-1}, \vec{a}_j, \vec{e}_{r+1}, \dots, \vec{e}_m]$

& $A = B_2 Y_2$

What is Y_2 ?

$$(1) \Rightarrow \quad A = C\, [\vec{u}_1, \vec{u}_2 \cdots \vec{u}_{j-1}, \vec{e}_r, \cdots \vec{u}_{n-m} \mid \vec{e}_1 \cdots \vec{u}_r \cdots \vec{e}_m] \quad (2)$$

$$\Leftrightarrow \begin{cases} \vec{a}_j = C\vec{e}_r \\ \vec{e}_1 = C\vec{e}_1 \\ \vdots \\ \vec{e}_m = C\vec{e}_m \end{cases} \Leftrightarrow \quad C = [\vec{e}_1, \vec{e}_2, \dots \vec{e}_{r-1}, \vec{a}_j, \vec{e}_{r+1}, \dots \vec{e}_m] = B_2 \quad (3)$$

$$(3) \,\&\, (2) \Rightarrow \quad A = B_2 \underbrace{[\vec{u}_1 \cdots \vec{u}_{j-1}, \vec{e}_r, \cdots u_{n-m} \mid \vec{e}_1 \cdots \vec{u}_r \cdots \vec{e}_m]}_{= Y_2}$$

#

Simplex Tableau

$$A\vec{x} = \vec{b}$$

$$[N \mid B]\begin{bmatrix}\vec{x}_N \\ \vec{x}_B\end{bmatrix} = \vec{b} \quad (1) \qquad \vec{x}_N = \vec{0} \quad (2)$$

$$[\vec{C}_N^T \mid \vec{C}_B^T]\begin{bmatrix}\vec{x}_N \\ \vec{x}_B\end{bmatrix} = x_0 \quad (3)$$

$$(1) \Rightarrow \quad \underbrace{[B^{-1}N \mid I]}_{Y}\begin{bmatrix}\vec{x}_N \\ \vec{x}_B\end{bmatrix} = B^{-1}\vec{b}$$

$$\Rightarrow \quad \vec{x}_B = B^{-1}\vec{b} - B^{-1}N\vec{x}_N \quad (4)$$

$$\overset{(2)}{\Rightarrow} \quad \vec{x}_B = B^{-1}\vec{b}$$

$$(3) \Rightarrow \quad \vec{C}_N^T\vec{x}_N + \vec{C}_B^T(B^{-1}\vec{b} - B^{-1}N\vec{x}_N) = x_0$$

$$\Rightarrow \quad (\vec{C}_N^T - \underbrace{\vec{C}_B^T B^{-1}N}_{\vec{z}_N^T})\vec{x}_N + \vec{C}_B^T B^{-1}\vec{b} = x_0 \quad (5)$$

$$\Rightarrow \quad x_0 = \underbrace{\vec{C}_B^T B^{-1}b}_{\text{Current value}} \quad (6)$$

Current value a constant. (does not depend on x_B or x_N)

Simplex Tableau

$$\begin{bmatrix} \overbrace{B^{-1}N \mid I}^{Y} \\ \hline -(\vec{C}_N^T - \vec{z}_N^T) \mid 0 \end{bmatrix}\begin{bmatrix}\vec{x}_N \\ \vec{x}_B\end{bmatrix} = \begin{bmatrix} B^{-1}\vec{b} \\ \hline \vec{C}_B^T B^{-1} b\end{bmatrix}$$

$\vec{x}_N = 0$; current $\vec{x}_B$; current x_0

Simplex Tableau:

$$\left[\begin{array}{c:c|c} \overbrace{B^{-1}N}^{\text{Nonbasic}} & \overbrace{I}^{\text{basic}} & B^{-1}\vec{b} \\ \hline -(\vec{C}_N^T - \vec{z}_N^T) & 0 & \vec{C}_B^T B^{-1}\vec{b} \end{array}\right]\begin{array}{l} \} \text{ Current solution} \\ \} \text{ Current } x_0 \end{array}$$

$$x_0 = 3x_1 + x_2 + 3x_3 + 0x_4 + 0x_5 + 0x_6$$

$$\begin{cases} 2x_1 + x_2 + x_3 + x_4 = 2 \\ x_1 + 2x_2 + 3x_3 + x_5 = 5 \\ 2x_1 + 2x_2 + x_3 + x_6 = 6 \\ x_1, x_2, x_3, x_4, x_5, x_6 \geq 0 \end{cases}$$

T_0:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 2 \\ 5 \\ 6 \end{pmatrix}$$

$$x_0 = 3x_1 + x_2 + 3x_3 + 0x_4 + 0x_5 + 0x_6$$

$$= 0$$

T_1:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ 0 \\ 0 \\ 1 \\ 2 \end{pmatrix}$$

$$x_0 = ?\,x_1 + 0x_2 + ?\,x_3 + ?\,x_4 + 0x_5 + 0x_6$$

express x_2 in terms of x_1, x_3 & x_4

$$x_2 = 2 - 2x_1 - x_3 - x_4$$

$$x_0 = 3x_1 + (3 - 2x_1 - x_3 - x_4) + 3x_3 + 0x_4 + 0x_5 + 0x_6$$

$$= \underbrace{x_1 + 0x_2 + 2x_3 - x_4 + 0x_5 + 0x_6}_{0} + 2$$

$$= 0 + 2$$